MAXIMUM INDEPENDENT SET IN PLANAR GRAPH

Presented by
Himanshu Dutta
Nonserial dynamic programming:

- Such a problem is of the following form: maximize the objective $f(x_1, \cdots, x_n)$, where $f$ is given as a sum of terms $f_k(\cdot)$, each of which is a function of only a subset of the variables.
• Maximum Independent set problem:
Let $G = (V, E)$ be an undirected graph.
For each vertex $v_i$, $1 \leq i \leq n$, let $x_i$ be an associated variable which can assume a value of either zero or one.

Let the objective function $f(x_1, x_2, \cdots, x_n)$ be defined by

$$f(x_1, x_2, \cdots, x_n) = \sum_{(v_i, u_j) \in E} f_e(x_i, x_j) + \sum_{i=1}^{n} x_i,$$

where $f_e(x_i, x_j) = -\infty$ if $x_i = x_j = 1$, $f_e(x_i, x_j) = 0$ otherwise.

Maximum independent set in $G$ corresponds to an assignment of values 0, 1 to $x_1, x_2, \cdots, x_n$ which maximizes $f$.

$x_i = 1$ means $x_i$ is in the independent set
$x_i = 0$ means $x_i$ is not in the independent set.
Maximum Independent Set:

nonserial dynamic programming problem can be solved in $2^{O(n)}$ time.

- For Planar Graph:
  problem can be solved in $2^{O(\sqrt{n})}$ time.

  no term $f_k(\cdot)$ of $f$ can contain more than four variables, since the complete graph on five vertices is not planar.
The restriction of objective function:

\[ f = \sum_{k=1}^{m} f_k \] to a set of variables \( x_{i_1}, \cdots, x_{i_j} \) is the objective function \( f' = \sum \{ f_k | f_k \text{ depends only upon } x_{i_1}, \cdots, x_{i_j} \} \).

Lipton Tarjan Algorithm:

Given an objective function \( f(x_1, \cdots, x_n) = \sum_{k=1}^{m} f_k \) and a subset \( S \) of the variables \( x_1, \cdots, x_n \) which are constrained to have specific values, maximize \( f \) subject to the constraints on the variables in \( S \).
Lipton – Tarjan Exact Algorithm:

Step 1. If \( n < 100 \), solve the problem by exhaustively trying all possible assignments to the unconstrained variables. Otherwise, go to Step 2.

Step 2. Apply Corollary 1 to the interaction graph \( G \) of \( f \). Let \( A, B, C \) be the resulting vertex partition. Let \( f_1 \) be the restriction of \( f \) to \( A \cup C \) and let \( f_2 \) be the restriction of \( f \) to \( B \cup C \). For each possible assignment of values to the variables in \( C - S \), perform the following steps:

(a) Maximize \( f_1 \) with the given values for the variables in \( C \cup S \) by applying the method recursively;

(b) maximize \( f_2 \) with the given values for the variables in \( C \cup S \) by applying the method recursively;

(c) combine the solutions to (a) and (b) to obtain a maximum value of \( f \) with the given values for the variables in \( C \cup S \).

Choose the assignment of values to variables in \( C \cup S \) which maximizes \( f \) and return the appropriate value of \( f \) as the solution.

If \( n \geq 100 \), the algorithm solves at most \( 2^{O(\sqrt{n})} \) subproblems in Step 2, since \( C \) is of \( O(\sqrt{n}) \) size.

Each subproblem contains at most \( 2n/3 + 2\sqrt{2}\sqrt{n} \leq 29n/30 \) variables.

\( t(n) \leq O(n) + 2^{O(\sqrt{n})} \cdot t(29n/30) \) if \( n \geq 100 \), \( t(n) = O(1) \) if \( n < 100 \).
Approximate algorithm:

- **THEOREM:** Let $G$ be an $n$-vertex planar graph with nonnegative vertex costs summing to no more than one and let $0 \leq \varepsilon \leq 1$. Then there is some set $S$ of $O(\sqrt{n/\varepsilon})$ vertices whose removal leaves $G$ with no connected component of cost exceeding $\varepsilon$. Furthermore the set $C$ can be found in $O(n \log n)$ time.
Lipton – Tarjan Algorithm to find approximate maximum Independent Set

**Step 1.** Apply Theorem to $G$ with $\varepsilon = k(n)/n$ and each vertex having cost $1/n$ to find a set of vertices $C$ of size $O(n/\sqrt{k(n)})$ whose removal leaves no connected component with more than $k(n)$ vertices.

**Step 2.** In each connected component of $G$ minus $C$, find a maximum independent set by checking every subset of vertices for independence. Form $I$ as a union of maximum independent sets, one from each component.

Let $I^*$ be a maximum independent set of $G$. The restriction of $I^*$ to one of the connected components formed when $C$ is removed from $G$ can be no larger than the restriction of $I$ to the same component. Thus $|I^*| - |I| = O(n/\sqrt{k(n)})$. Since $G$ is planar, $G$ is four-colorable, and $|I^*| \geq n/4$. Thus $(|I^*| - |I|)/|I^*| = O(1/\sqrt{k(n)})$, and the relative error in the size of $I$ tends to zero with increasing $n$ as long as $k(n)$ tends to infinity with increasing $n$.

Step 1 of the algorithm requires $O(n \log n)$ time by Theorem 2. Step 2 requires $O(n_i 2^{n_i})$ time on a connected component of $n_i$ vertices. The total time required by Step 2 is thus

$$O\left(\max \left\{ \sum_{i=1}^{n} n_i 2^{n_i} \right| \sum_{i=1}^{n} n_i = n \text{ and } \frac{n}{k(n)} \leq n_i \leq k(n) \right) = O\left(\frac{n}{k(n)} k(n) 2^{k(n)}\right) = O(n 2^{k(n)}).$$

Hence the entire algorithm requires $O(n \cdot \max \{|\log n, 2^{k(n)}|\})$ time. If we choose $k(n) = \log n$, we get an $O(n^2)$-time algorithm with $O(1/\sqrt{\log n})$ relative error. If we choose $k(n) = \log \log n$, we get an $O(n \log n)$ algorithm with $O(1/\sqrt{\log \log n})$ relative error.
Planar Separator Theorem by Lipton and Tarjan

**THEOREM:**

- Let $G$ be any $n$-vertex planar graph with nonnegative vertex costs summing to no more than one. Then the vertices of $G$ can be partitioned into three sets $A$, $B$, $S$, such that no edge joins a vertex in $A$ with a vertex in $B$, neither $A$ nor $B$ has total vertex cost exceeding $2/3$, and $S$ contains no more than $2\sqrt{2}\sqrt{n}$ vertices. Furthermore $A$, $B$, $S$ can be found in $O(n)$ time.

For equal cost vertices, this becomes:

- Let $G$ be any $n$-vertex planar graph. Then the vertices of $G$ can be partitioned into three sets $A$, $B$, $S$, such that no edge joins a vertex in $A$ with a vertex in $B$, neither $A$ nor $B$ contains more than $2n/3$ vertices, and $S$ contains no more than $2\sqrt{2}\sqrt{n}$ vertices.
Planar Separator Theorem:

- Lemma 1: Let \( G \) be any planar graph. Shrinking any edges of \( G \) to a single vertex preserves planarity.

- Corollary 1. Let \( G \) be any planar graph. Shrinking any connected subgraph of \( G \) to a single vertex preserves planarity.

- Lemma 2: Let \( G \) be any planar graph with non-negative vertex costs summing to no more than one. Suppose \( G \) has a spanning tree of radius \( r \). Then the vertices of \( G \) can be partitioned into three sets \( A, B, C \), such that no edge joins a vertex in \( A \) with a vertex in \( B \), neither \( A \) nor \( B \) has total cost exceeding \( 2/3 \), and \( C \) contains no more than \( 2r + 1 \) vertices, one the root of the tree.

- Lemma 3: Let \( G \) be any \( n \)-vertex connected planar graph having nonnegative vertex costs summing to no more than one. Suppose that the vertices of \( G \) are partitioned into levels according to their distance from some vertex \( v \), and that \( L(l) \) denotes the number of vertices on level \( l \). If \( r \) is the maximum distance of any vertex from \( v \), let \( r + 1 \) be an additional level containing no vertices. Given any two levels \( l_1 \) and \( l_2 \) such that levels 0 through \( l_1 - 1 \) have total cost not exceeding \( 2/3 \) and levels \( l_2 + 1 \) through \( r + 1 \) have total cost not exceeding \( 2/3 \), it is possible to find a partition \( A, B, C \) of the vertices of \( G \) such that no edge joins a vertex in \( A \) with a vertex in \( B \), neither \( A \) nor \( B \) has total cost exceeding \( 2/3 \), and \( C \) contains no more than \( L(l_1) + L(l_2) + \max \{0, 2(l_2 - l_1 - 1)\} \) vertices.
Proof. Assume $G$ is connected. Partition the vertices into levels according to their
distance from some vertex $v$. Let $L(l)$ be the number of vertices on level $l$. If $r$ is the
maximum distance of any vertex from $v$, define additional levels $-1$ and $r + 1$ containing
no vertices.

Let $l_1$ be the level such that the sum of costs in levels 0 through $l_1 - 1$ is less
than $1/2$, but the sum of costs in levels 0 through $l_1$ is at least $1/2$. (If no such $l_1$ exists,
the total cost of all vertices is less than $1/2$, and $B = C = \emptyset$ satisfies the theorem.) Let
$k$ be the number of vertices on levels 0 through $l_1$. Find a level $l_0$ such that $l_0 \leq l_1$ and
$|L(l_0)| + 2(l_1 - l_0) \leq 2\sqrt{k}$. Find a level $l_2$ such that $l_1 + 1 \leq l_2$ and $|L(l_2)| + 2(l_2 - l_1 - 1) \leq 2\sqrt{n - k}$. If two such levels exist, then by Lemma 3 the vertices of $G$
can be partitioned into three sets $A$, $B$, $C$ such that no edge joins a vertex in $A$ with a vertex in $B$, neither
$A$ nor $C$ has cost exceeding $2/3$, and $C$ contains no more than $2(\sqrt{k} + \sqrt{n - k})$ vertices.
But $2(\sqrt{k} + \sqrt{n - k}) \leq 2(\sqrt{n/2} + \sqrt{n/2}) = 2\sqrt{2n}$. Thus the theorem holds if suitable
levels $l_0$ and $l_2$ exist.

Suppose a suitable level $l_0$ does not exist. Then, for $i \leq l_1$, $L(i) \geq 2\sqrt{k} - 2(l_1 - i)$.
Since $L(0) = 1$, this means $1 \geq 2\sqrt{k} - 2l_1$, and $l_1 + 1/2 \geq \sqrt{k}$. Thus $l_1 = [l_1 + 1/2] \geq
[\sqrt{k}]$, and

$$k = \sum_{i=0}^{l_1} L(i) \geq \sum_{i=l_1-[\sqrt{k}]}^{l_1} 2\sqrt{k} - 2(l_1 - i) \geq 4\sqrt{k} - 2[\sqrt{k}])([\sqrt{k}] + 1)/2 \geq \sqrt{k}([\sqrt{k}] + 1) > k.$$

This is a contradiction. A similar contradiction arises if a suitable level $l_2$ does not
exist. This completes the proof for connected graphs.
Now suppose $G$ is not connected. Let $G_1, G_2, \cdots, G_k$ be the connected components of $G$, with vertex sets $V_1, V_2, \cdots, V_k$, respectively. If no connected component has total vertex cost exceeding $1/3$, let $i$ be the minimum index such that the total cost of $V_1 \cup V_2 \cup \cdots \cup V_i$ exceeds $1/3$. Let $A = V_1 \cup V_2 \cup \cdots \cup V_i$, let $B = V_{i+1} \cup V_{i+2} \cup \cdots \cup V_k$, and let $C = \emptyset$. Since $i$ is minimum and the cost of $V_i$ does not exceed $1/3$, the cost of $A$ does not exceed $2/3$. Thus the theorem is true.

If some connected component (say $G_i$) has total vertex cost between $1/3$ and $2/3$, let $A = V_i$, $B = V_1 \cup \cdots \cup V_{i-1} \cup V_{i+1} \cup \cdots \cup V_k$, and $C = \emptyset$. Then the theorem is true.

Finally, if some connected component (say $G_i$) has total vertex cost exceeding $2/3$, apply the above argument to $G_i$. Let $A^*, B^*, C^*$ be the resulting partition. Let $A$ be the set among $A^*$ and $B^*$ with greater cost, let $C = C^*$, and let $B$ be the remaining vertices of $G$. Then $A$ and $B$ have cost not exceeding $2/3$ and the theorem is true.

This proves the theorem for all planar graphs. In all cases the separator $C$ is either empty or contained in only one connected component of $G$. □
References:
