PLANAR SEPARATOR AND GEOMETRIC EXTENSIONS

Presented by
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Preliminaries:

- **Why Study Separation**: Separation is a fundamental tool in problem solving using divide and conquer paradigm. Good separators can be used to obtain exact sub-exponential time algorithms, or polynomial time approximation algorithms for NP-hard problems.

- **Vertex Separator**: Let $G = (V, E)$ be a graph. Let $(A, B, S)$ be a non-trivial partition of $V$ so that there are no edges between $A$ and $B$. Then $S$ is called a Vertex Separator.

- **Packing**: Finding the maximum subcollection of pairwise disjoint objects.
  Packing rectangles in plane has application in VLSI.

- **Piercing**: Finding the minimum point set that intersects every object.
  One of its applications of piercing is in the area of locating emergency facilities such that all potential customers will be within a reasonably small radius around the facility.

  piercing number $\geq$ packing number
- **Planar graphs**
  A planar graph is a graph that can be embedded in the plane (or in a sphere) in such a way that its edges intersect only at their endpoints.

A planar Graph

A non-planar Graph
Planar Separator Theorem by Lipton and Tarjan

THEOREM 1 (L-T):

- Let $G$ be any $n$-vertex planar graph with nonnegative vertex costs summing to no more than one. Then the vertices of $G$ can be partitioned into three sets $A$, $B$, $S$, such that no edge joins a vertex in $A$ with a vertex in $B$, neither $A$ nor $B$ has total vertex cost exceeding $2/3$, and $S$ contains no more than $2\sqrt{2}\sqrt{n}$ vertices. Furthermore $A$, $B$, $S$ can be found in $O(n)$ time.

For equal cost vertices, this becomes:

- Let $G$ be any $n$-vertex planar graph. Then the vertices of $G$ can be partitioned into three sets $A$, $B$, $S$, such that no edge joins a vertex in $A$ with a vertex in $B$, neither $A$ nor $B$ contains more than $2n/3$ vertices, and $S$ contains no more than $2\sqrt{2}\sqrt{n}$ vertices.
• Basically for planar graph, we can find Separator of size $O(\sqrt{n})$ in time $O(n)$.
• In the graph below, Number of vertices = 40
• According to Lipton – Tarjan Theorem, we can find a separator of size no more than $2\sqrt{2}\sqrt{40}$

Vertex Separator

Reference:
A Separator Theorem for Planar Graphs – Lipton – Tarjan
http://en.wikipedia.org/wiki/Vertex_separator
Before Lipton and Tarjan Theorem:
Previously known separator theorems included the following:

• Any n-vertex binary tree can be separated into two subtrees, each with no more than $\frac{2n}{3}$ vertices, by removing a single edge.

• Any n-vertex binary tree can be divided into two parts, each with no more than $\frac{2n}{3}$ vertices, by removing a single vertex.

• $\sqrt{n}$ separator theorem holds for the class of grid graphs.
Proof of Lipton- Tarjan involves following 3 well known theorem

- **Theorem A [Jordan Curve Theorem]**: Let $C$ be any closed curve in the plane. Removal of $C$ divides the plane into exactly two connected regions, the "inside" and the "outside" of $C$.

- **Theorem B**: Any $n$-vertex planar graph with $n \geq 3$ contains no more than $3n - 6$ edges.

- **Theorem C [Kuratowski Theorem]**: A graph is planar if and only if it contains neither a complete graph on five vertices nor a complete bipartite graph on two sets of three vertices as a generalized subgraph.
Illustration of Jordan curve theorem

Reference:
Illustration of Kuratowski Theorem

Reference: A Separator Theorem for Planar Graphs – Lipton – Tarjan
From Kuratowski Theorem:

- Lemma 1: Let $G$ be any planar graph. Shrinking any edges of $G$ to a single vertex preserves planarity.

- Corollary 1. Let $G$ be any planar graph. Shrinking any connected subgraph of $G$ to a single vertex preserves planarity.

- Lemma 2: Let $G$ be any planar graph with non-negative vertex costs summing to no more than one. Suppose $G$ has a spanning tree of radius $r$. Then the vertices of $G$ can be partitioned into three sets $A$, $B$, $C$, such that no edge joins a vertex in $A$ with a vertex in $B$, neither $A$ nor $B$ has total cost exceeding $\frac{2}{3}$, and $C$ contains no more than $2r + 1$ vertices, one the root of the tree.

Note: A spanning tree of a graph is a subset of edges that form a tree.
Shrinking an edge to form a Kuratowski graph. Original graph must contain a Kuratowski graph as a generalized subgraph.

Reference: A Separator Theorem for Planar Graphs – Lipton – Tarjan
APPLICATIONS OF A PLANAR SEPARATOR THEOREM: Lipton Tarjan

- **Lemma**: Let $G$ be any $n$-vertex graph of genus $g > 0$. Then there exists a subset of no more than $\sqrt{2n}$ vertices whose removal reduces the genus of $G$ by at least one.

- **THEOREM 2 (L-T)**: If $G$ is an $n$-vertex graph of genus $g > 0$, there is subset of no more than $g\sqrt{2n}$ vertices whose removal leaves a planar graph.
Some algorithmic applications:

- **THEOREM:** The maximum independent set in any planer graph can be computed in $2^{O(\sqrt{n})}$ time.

- **THEOREM 3 (L-T):** Let $G$ be an $n$-vertex planar graph with nonnegative vertex costs summing to no more than one and let $0 \leq \varepsilon \leq 1$. Then there is some set $S$ of $O(\sqrt{n}/\varepsilon)$ vertices whose removal leaves $G$ with no connected component of cost exceeding $\varepsilon$. Furthermore the set $C$ can be found in $O(n \log n)$ time.

- Lipton – Tarjan used above Theorem to find an approximately maximum independent set in a planar graph $G = (V, E)$. 

![Maximum Independent Set](image)
GEOMETRIC EXTENSIONS
Intersection Graph

- An intersection graph is a graph that represents the pattern of intersections of a family of sets. Any graph may be represented as an intersection graph, but some important special classes of graphs may be defined by the types of sets that are used to form an intersection representation of them.

- Formally, an intersection graph is an undirected graph formed from a family of sets $S_i$, $i = 0, 1, 2, \ldots$ by creating one vertex $v_i$ for each set $S_i$, and connecting two vertices $v_i$ and $v_j$ by an edge whenever the corresponding two sets have a nonempty intersection, that is, $E(G) = \{v_i, v_j\} \mid S_i \cap S_j \neq \emptyset$. 
Koebe’s Theorem: Every planar graph can be realized as contact graph of set of touching disks (of possibly varying size). More precisely, a graph is planar iff it is the “contact” graph of a set of interior-disjoint disks.

Extensions to non-planar case [Chan 2003]

• Chan derived a separation theorem with respect to a clique measure for intersection graph of fat objects, i.e., spheres, cubes, etc.

• A collection $C$ of objects is fat if for any $r$ and size-$r$ box $R$, we can choose a constant number $c$ of points such that every object that intersects $R$ and has size at least $r$ contains one of the chosen points.
Timothy Chan:

- Chan described an algorithm for fat objects packing, based on geometric separators, that requires only linear space. This algorithm can also be applied to piercing, yielding the first PTAS for that problem.

- Using separator approach Chan gave an $1 + \varepsilon$ - approximation, for a collection of fat objects in $n^{O(1/\varepsilon^d)}$ time and linear space.

Chan’s Separation Theorem for $\mathbb{R}^2$:

- Given a collection of $n$ disjoint fat objects in $\mathbb{R}^2$, there exists a box $R$ (rectangle or circle etc.) such that $2n/3$ objects are inside box $R$, at most $2n/3$ objects are outside box and at most $O(\sqrt{n})$ objects intersect the boundary of the box $R$. 
Chan’s Separation Theorem for $\mathbb{R}^d$:

- Given a collection of $n$ disjoint fat objects in $\mathbb{R}^d$, there exists a box $R$ such that $2n/3$ objects are inside $R$, at most $2n/3$ objects are outside $R$ and at most $O(n^{1-1/d})$ objects intersect the boundary of $R$. \

On $d^{1/()}dOn$
Dr. Shahrokhi:

- **Clique cover**: Given a graph $G = (V, E)$, let $\rho = \{p_1, p_2, \ldots, p_i\}$ be a partition of $V$. $\rho$ is called clique cover for $G$, if induced subgraph $p_i; i = 1, 2, \ldots, k$, is clique in $G$.

- **Clique cover width**: The clique cover width of $G$, denoted by $CCW(G)$, is the minimum value of the bandwidth of all clique cover graphs in $G$. 
One way placing the vertices of above graph into the integer points. Maximum length of edge, bd = 4 − 1 = 3

Bandwidth = 4 − 2 = 2
**Clique Cover with 3 Cliques**

Note: dotted red lines show the clique in the graph

**Clique cover graph**

Note: Clique cover graph is obtained by contracting the vertices of each clique into a single vertex.

Clique cover width = 3 - 1 = 2

**Clique Cover with 4 Cliques**

Clique cover width = 1
Connections between clique cover width, bandwidth and clique separators [Shahrokhi 2010]:

- The smaller the bandwidth, the smaller is the separator

- If the graph is dense, then it does not have small separators. However, in this case we may be able get a balanced vertex separation of the clique cover graph, and hence the original graph [Shahrokhi 2010]

- Let $C$ be a clique cover of $G$, and let $L: C_1, C_2, \ldots, C_K$, $K \geq 2$ be a linear ordering of cliques in $C$. Then, there is a separation $(U, W, T)$ in $G$ so that $\beta(G[T]) \leq W(L)$, $\beta(G[U]) \leq \frac{2|C|}{3}$, and $\beta(G[W]) \leq \frac{2|C|}{3}$ [Shahrokhi 2010]

For a subgraph $H$ of $G$, $\beta(H)$ denotes the clique cover number of $H$. 
[Shahrokhi 2010]:

- Let $G = (V(G), E(G))$ be the intersection graph of a set of axis parallel unit height rectangles in the plane. Then, a maximum independent set in $G$ can be computed in $|V(G)|^{O(\sqrt{\alpha(G)})}$, where $\alpha(G)$ is the independence number of $G$. Moreover, there is a PTAS that gives a $(1 - \varepsilon)$-approximate solution to $\alpha(G)$ in $|V(G)|^{O(\frac{1}{\varepsilon})}$ time and requires $O(|V(G)|)^2$ storage.

- Let $S$ be a set of axis parallel unit height rectangles in the plane. Then, the piercing number of $S$ can be computed in $|S|^{O(\sqrt{P(S)})}$, where $P(S)$ is the piercing number of $S$. Moreover, there is a PTAS that gives a $(1 + \varepsilon)$-approximate solution to $P(S)$ in $|S|^{O(\frac{1}{\varepsilon})}$ time and requires $O(|S|)^2$ storage.
References:


