Transfer Functions

- Transfer functions of CT systems can be found from analysis of
  - Differential Equations
  - Block Diagrams
  - Circuit Diagrams

A circuit can be described by a system of differential equations:

\[
R_1 i_1(t) + L \frac{d}{dt}(i_1(t)) - \frac{d}{dt}(i_2(t)) = v(t)
\]

\[
L \frac{d}{dt}(i_2(t)) - \frac{d}{dt}(i_1(t)) + \frac{1}{C} \int i_2(t) dt + v(t) + R_1 i_2(t) = 0
\]

Using the Laplace transform, a circuit can be described by a system of algebraic equations:

\[
R_1 I_1(s) + sL I_1(s) - sLI_2(s) = V_g(s)
\]

\[
sLI_2(s) - sLI_1(s) + \frac{1}{C} I_2(s) + R_2 I_2(s) = 0
\]

A mechanical system can be described by a system of differential equations:

\[
\ddot{x}(t) - K_x x(t) - K_s \dot{x}(t) - x(t) = m \ddot{y}(t)
\]

\[
K_s \dot{x}(t) - x(t) - K_s \dot{x}(t) = m_s \ddot{y}(t)
\]

A mechanical system can also be described by a system of algebraic equations:

\[
\begin{align*}
W_s X_s(s) - K_s X_s(s) - X_s(s) &= m_s X_s(s) \\
K_s X_s(s) - x(t) - K_s \dot{x}(t) &= m_s \ddot{y}(t)
\end{align*}
\]

or a system of algebraic equations:

\[
\begin{align*}
W_s X_s(s) - K_s X_s(s) - X_s(s) &= m_s X_s(s) \\
W_s X_s(s) - K_s X_s(s) - X_s(s) &= m_s X_s(s)
\end{align*}
\]
Transfer Functions

The mechanical system can also be described by a block diagram.

Time Domain

Frequency Domain

System Stability

- System stability is very important
- A continuous-time LTI system is stable if its impulse response is absolutely integrable
- This translates into the frequency domain as the requirement that all the poles of the system transfer function must lie in the open left half-plane of the s plane (pp. 675-676)
- “Open left half-plane” means not including the \( \omega \) axis

System Interconnections

- **Cascade**
  
  \[
  X(s) \rightarrow H_1(s) \rightarrow X(s)H_2(s) \rightarrow Y(s) = X(s)H_1(s)H_2(s)
  \]

- **Parallel**
  
  \[
  X(s) \rightarrow H_1(s)H_2(s) \rightarrow Y(s)
  \]

Analysis of Feedback Systems

**Beneficial Effects**

\[
Y(s) = \frac{K}{1 + KH_2(s)}
\]

If \( K \) is large enough that \( KH_2(s) \gg 1 \) then \( H(s) = \frac{1}{KH_2(s)} \). This means that the overall system is the approximate inverse of the system in the feedback path. This kind of system can be useful for reversing the effects of another system.

Analysis of Feedback Systems

A very important example of feedback systems is an electronic amplifier based on an operational amplifier

\[
H_1(s) = \frac{V_i(s)}{V_o(s)} = \frac{A_o}{1 + \frac{s}{p}}
\]

Let the operational amplifier gain be

\[
H_1(s) = \frac{V_i(s)}{V_o(s)} = \frac{A_o}{1 + \frac{s}{p}}
\]
Analysis of Feedback Systems

The amplifier can be modeled as a feedback system with this block diagram.

\[
\frac{V_0(s)}{V_i(s)} = \frac{-A_0 Z_f(s)}{1 - \frac{Z_f(s)}{Z_i(s)}}
\]

The overall gain can be written as

\[
\frac{V_0(s)}{V_i(s)} = \frac{-A_0 Z_f(s)}{1 - \frac{Z_f(s)}{Z_i(s)}}
\]

If the operational amplifier low-frequency gain, \( A_0 \), is very large (which it usually is) then the overall amplifier gain reduces at low-frequencies to

\[
\frac{V_0(s)}{V_i(s)} = \frac{Z_f(s)}{Z_i(s)}
\]

the gain formula based on an ideal operational amplifier.

Feedback can stabilize an unstable system. Let a forward-path transfer function be

\[
H_1(s) = \frac{1}{s-p}, \quad p > 0
\]

This system is unstable because it has a pole in the right half-plane. If we then connect feedback with a transfer function, \( K \), a constant, the overall system gain becomes

\[
H(s) = \frac{1}{s-p + K}
\]

and, if \( K > p \), the overall system is now stable.

Feedback can make an unstable system stable but it can also make a stable system unstable. Even though all the poles of the forward and feedback systems may be in the open left half-plane, the poles of the overall feedback system can be in the right half-plane.

A familiar example of this kind of instability caused by feedback is a public address system. If the amplifier gain is set too high the system will go unstable and oscillate, usually with a very annoying high-pitched tone.
Stable Oscillation Using Feedback

Prototype Feedback System

Feedback System Without Excitation

Can the response be non-zero when the excitation is zero? Yes, if the overall system gain is infinite. If the system transfer function has a pole pair on the $\omega$ axis, then it is infinite at the frequency of that pole pair and there can be a response without an excitation. In practical terms the trick is to be sure the poles stay on the $\omega$ axis. If the poles move into the left half-plane the response attenuates with time. If the poles move into the right half-plane the response grows with time (until the system goes non-linear).

A real example of a system that oscillates stably is a laser. In a laser the forward path is an optical amplifier. Laser action begins when a photon is spontaneously emitted from the pumped medium in a direction normal to the mirrors. If the "round-trip" gain of the combination of pumped laser medium and mirrors is unity, sustained oscillation of light will occur. For that to occur the wavelength of the light must fit into the distance between mirrors an integer number of times.

A laser can be modeled by a block diagram in which the $K$s represent the gain of the pumped medium or the reflection or transmission coefficient at a mirror, $L$ is the distance between mirrors and $c$ is the speed of light.
**Analysis of Feedback Systems**

The Routh-Hurwitz Stability Test

The Routh-Hurwitz Stability Test is a method for determining the stability of a system if its transfer function is expressed as a ratio of polynomials in $s$. Let the numerator be $N(s)$ and let the denominator be

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n$$

The first two rows contain the coefficients of the denominator polynomial. The entries in the following row are found by the formulas,

$$b_{p-2} = \frac{a_{p-2} a_{p-1}}{a_{p-1}}$$

$$b_{p-4} = \frac{a_{p-4} a_{p-3}}{a_{p-3}}$$

The entries on succeeding rows are computed by the same process based on previous row entries. If there are any zeros or sign changes in the $a_{k1}$ column, the system is unstable. The number of sign changes in the column is the number of poles in the right half-plane (pp. 693-694).

**Root Locus**

Common Type of Feedback System

$$X(s) \quad + \quad K \quad \frac{H(s)}{1 + K H_1(s) H_2(s)} \quad Y(s)$$

System Transfer Function

$$H(s) = \frac{K H_1(s)}{1 + K H_1(s) H_2(s)}$$

Loop Transfer Function

$$T(s) = K H_1(s) H_2(s)$$

$K$ can range from zero to infinity. For $K$ approaching zero, using

$$Q(s) + K P(s) = 0$$

the poles of $H$ are the same as the zeros of $Q(s) = 0$ which are the poles of $T$. For $K$ approaching infinity, using

$$\frac{Q(s)}{K} + P(s) = 0$$

the poles of $H$ are the same as the zeros of $P(s) = 0$ which are the zeros of $T$. So the poles of $H$ start on the poles of $T$ and terminate on the zeros of $T$, some of which may be at infinity. The curves traced by these pole locations as $K$ is varied are called the root locus.
Let $H_1(s) = \frac{K}{s^3 + 2s^2 + 3}$ and let $H_2(s) = 1$.

Then $T(s) = \frac{K}{s^3 + 2s^2 + 3}$.

No matter how large $K$ gets this system is stable because the poles always lie in the left half-plane.

At some finite value of $K$ the system becomes unstable because two poles move into the right half-plane.

### Four Rules for Drawing a Root Locus

1. Each root-locus branch begins on a pole of $T$ and terminates on a zero of $T$.

2. Any portion of the real axis for which the sum of the number of real poles and/or real zeros lying to its right is odd, is a part of the root locus.

3. The root locus is symmetrical about the real axis.

4. If the number of poles of $T$ exceeds the number of zeros of $T$ by an integer, $m$, then $m$ branches of the root locus terminate on zeros of $T$ which lie at infinity. Each of these branches approaches a straight-line asymptote and the angles of these asymptotes are at the angles, $\frac{\pi k}{m}$, $k = 1, 3, 5, ...$ with respect to the positive real axis. These asymptotes intersect on the real axis at the location, $\sigma = \frac{1}{m} \left( \sum \text{finite poles} - \sum \text{finite zeros} \right)$.

### Gain and Phase Margin

Real systems are usually designed with a margin of error to allow for small parameter variations and still be stable.

That “margin” can be viewed as a gain margin or a phase margin.

System instability occurs if, for any real $\omega$,

$$T(j\omega) = 1 - \text{a number with a magnitude of one and a phase of } -\pi \text{ radians.}$$
Analysis of Feedback Systems

Gain and Phase Margin

So to be guaranteed stable, a system must have a $T$ whose magnitude, as a function of frequency, is less than one when the phase hits $-\pi$ or, seen another way, $T$ must have a phase, as a function of frequency, more positive than $-\pi$ for all $|T|$ greater than one.

The difference between the magnitude of $T$ of 0 dB and the magnitude of $T$ when the phase hits $-\pi$ is the gain margin.

The difference between the phase of $T$ when the magnitude hits 0 dB and a phase of $-\pi$ is the phase margin.

Steady-State Tracking Errors in Unity-Gain Feedback Systems

A very common type of feedback system is the unity-gain feedback connection.

$$X(s) \rightarrow E(s) \rightarrow H(s) \rightarrow Y(s)$$

The aim of this type of system is to make the response “track” the excitation. When the error signal is zero, the excitation and response are equal.

The steady-state value of this signal is (using the final-value theorem)

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} s X(s) = \lim_{s \to 0} s (1 + H(s)) = \frac{X(s)}{1 + H(s)}$$

If the excitation is the unit step, $A u(t)$, then the steady-state error is

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} s X(s) = \lim_{s \to 0} s \frac{A}{1 + H(s)}$$

If $a_0 = 0$ and $b_1 \neq 0$, the steady-state error is zero and the forward transfer function can be written as

$$H(s) = \frac{b_0 s^2 + b_1 s + b_2}{a_0 s^2 + a_1 s + a_2}$$

which has a pole at $s = 0$.

Gain and Phase Margin

Steady-State Tracking Errors in Unity-Gain Feedback Systems

The Laplace transform of the error signal is

$$E(s) = \frac{X(s)}{1 + H(s)}$$

The steady-state value of this signal is (using the final-value theorem)

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} s E(s) = \lim_{s \to 0} \frac{X(s)}{1 + H(s)}$$

If $a_0 = 0$ and $b_1 \neq 0$, then the steady-state error is zero and the forward transfer function can be written as

$$H(s) = \frac{b_0 s^2 + b_1 s + b_2}{a_0 s^2 + a_1 s + a_2}$$

which has a pole at $s = 0$.

Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

If the forward transfer function of a unity-gain feedback system has a pole at zero and the system is stable, the steady-state error with step excitation is zero. This type of system is called a “type 1” system (one pole at $s = 0$ in the forward transfer function). If there are no poles at $s = 0$, it is called a “type 0” system and the steady-state error with step excitation is non-zero.

$$h(t)^{(0)}$$ Type 0 System

$$h(t)^{(0)}$$ Type 1 System
Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

The steady-state error with ramp excitation is:

- Infinite for a stable type 0 system
- Finite and non-zero for a stable type 1 system
- Zero for a stable type 2 system (2 poles at $s = 0$ in the forward transfer function)

Block Diagram Reduction

It is possible, by a series of operations, to reduce a complicated block diagram down to a single block.

Moving a Pick-Off Point

Combining Two Summers

Move Pick-Off Point

Move Summer
Block Diagram Reduction

**Combine Summers**

- Diagram showing the combination of summers in a block diagram.

**Combine Parallel Blocks**

- Diagram illustrating the combination of parallel blocks in a block diagram.

**Combine Cascaded Blocks**

- Diagram depicting the combination of cascaded blocks in a block diagram.

**Reduce Feedback Loop**

- Diagram for reducing feedback loops in a block diagram.

---

**Mason’s Theorem**

**Definitions:**

- Number of Paths from Input to Output - $N_p$
- Number of Feedback Loops - $N_L$
- Transfer Function of $i$th Path from Input to Output - $P_i(s)$
- Loop transfer function of $i$th Feedback Loop - $T_i(s)$

**Mason’s Theorem Formula:**

\[
\Delta(s) = 1 + \sum_{i=1}^{N_p} T_i(s) + \sum_{i=1}^{N_L} T_i(s)T_i^*(s) + \sum_{i=1}^{N_L} T_i(s)T_j(s)T_k(s) + \cdots
\]
Mason’s Theorem

The overall system transfer function is

\[ H(s) = \sum_{i=1}^{N} \frac{P_i(s)\Lambda_i(s)}{\Delta(s)} \]

where \( \Lambda_i(s) \) is the same as \( \Delta(s) \) except that all feedback loops which share a signal with the \( i \)th path, \( P_i(s) \), are excluded.

System Responses to Standard Signals

Unit Step Response

Let \( H(s) = \frac{N(s)}{D(s)} \) be proper in \( s \). Then the Laplace transform of the unit step response is

\[ Y(s) = H_u(s) = \frac{N(s)}{sD(s)} + \frac{K}{s} \quad K = H(0) \]

If the system is stable, the inverse Laplace transform of the response is

\[ y(t) = H(0) + \sum_{i=1}^{N} \frac{P_i(s)\Lambda_i(s)}{\Delta(s)} \]

is called the transient response and the steady-state response is

\[ y(t) = \frac{H(0)}{s} . \]

System Responses to Standard Signals

Unit Step Response

Let \( H(s) = \frac{N(s)}{D(s)} \) be proper in \( s \). If the Laplace transform of the excitation is some general excitation, \( X(s) \), then the Laplace transform of the response is

\[ Y(s) = \frac{N(s)}{D(s)} X(s) = \frac{N(s)N_u(s)}{D(s)D_u(s)} \]

same poles as system

\[ \frac{N(s)}{D(s)} \]

same poles as excitation

\[ N_u(s) \]
System Responses to Standard Signals

Unit Step Response

Let \( H(s) = \frac{A \omega_0}{s^2 + 2 \omega_0 \omega \omega + \omega_0^2} \)

\[ y(t) = h(t) \]

\[ h(t) = 0.5 \left( 1 - \frac{1}{e^{-\frac{t}{\omega}} + 0.5} \right) \]

\[ \omega_0 = 0.5 \]

\[ \omega = 0.2 \]

\[ \omega = 1 \]

\[ s \]

\[ t \]

Pole-Zero Diagrams and Frequency Response

If the transfer function of a system is \( H(s) \), the frequency response is \( H(j\omega) \). The most common type of transfer function is of the form:

\[ H(s) = A \left( \frac{s - z_1}{s - p_1} \right) \left( \frac{s - z_2}{s - p_2} \right) \]

Therefore \( H(j\omega) \) is:

\[ H(j\omega) = A \left( \frac{j\omega - z_1}{j\omega - p_1} \right) \left( \frac{j\omega - z_2}{j\omega - p_2} \right) \]

Pole-Zero Diagrams and Frequency Response

If the transfer function of a system is \( H(s) \), the frequency response is \( H(j\omega) \). The most common type of transfer function is of the form:

\[ H(s) = A \left( \frac{s - z_1}{s + p_1} \right) \left( \frac{s - z_2}{s + p_2} \right) \]

Therefore \( H(j\omega) \) is:

\[ H(j\omega) = A \left( \frac{j\omega - z_1}{j\omega + p_1} \right) \left( \frac{j\omega - z_2}{j\omega + p_2} \right) \]

Pole-Zero Diagrams and Frequency Response

Let \( H(s) = \frac{N(s)}{D(s)} \) be proper in \( s \). If the excitation is a suddenly-applied, unit-amplitude cosine, the response is:

\[ Y(s) = \frac{N(s)}{D(s)} \cdot \frac{s}{s^2 + \omega^2} \]

which can be reduced and inverse Laplace transformed into (pp. 713-714)

\[ y(t) = \mathcal{L}^{-1} \left( \frac{N(s)}{D(s)} \right) \cos(\omega t + \angle H(j\omega)) u(t) \]

If the system is stable, the steady-state response is a sinusoid of same frequency as the excitation but, generally, a different magnitude and phase.

System Responses to Standard Signals

Unit Step Response

Let \( H(s) = \frac{A \omega_0}{s^2 + 2 \omega_0 \omega s + \omega_0^2} \)

\[ y(t) = h(t) \]

\[ h(t) = 0.5 \left( 1 - \frac{1}{e^{-\frac{t}{\omega}} + 0.5} \right) \]

\[ \omega_0 = 0.5 \]

\[ \omega = 0.2 \]

\[ \omega = 1 \]

\[ s \]

\[ t \]
Butterworth Filters

The squared magnitude of an \( n \)th order, unity-gain, lowpass Butterworth filter with a corner frequency of 1 radian/s is

\[
|H(j\omega)|^2 = \frac{1}{1 + \omega^2}
\]

This is called a normalized Butterworth filter because its gain is normalized to one and its corner frequency is normalized to 1 radian/s.

A Butterworth filter transfer function has no finite zeros and the poles all lie on a semicircle in the left-half plane whose radius is the corner frequency in radians/s and the angle between the pole locations is always \( \frac{\pi}{n} \) radians.

Frequency Transformations

A normalized lowpass Butterworth filter can be transformed into an unnormalized highpass, bandpass or bandstop Butterworth filter through the following transformations (pp. 721-725).

- **Lowpass to Highpass**: 
  \[
  s \rightarrow \frac{\omega_0}{s}
  \]

- **Lowpass to Bandpass**: 
  \[
  s \rightarrow \frac{s^2 + \omega_0^2}{s^2 + \omega_n^2}
  \]

- **Lowpass to Bandstop**: 
  \[
  s \rightarrow \frac{s^2 + \omega_0^2}{s^2 + \omega_h^2}
  \]

Standard Realizations of Systems

There are multiple ways of drawing a system block diagram corresponding to a given transfer function of the form,

\[
H(s) = \frac{b_N s^N + \cdots + b_1 s + b_0}{s^N + a_N - \cdots + a_1 s + a_0}
\]

The system can then be realized in this form.

Canonical Form

The transfer function can be conceived as the product of two transfer functions,

\[
H_1(s) = \frac{Y_1(s)}{X(s)} = \frac{1}{s^N + a_N - \cdots + a_1 s + a_0}
\]

and

\[
H_2(s) = \frac{Y_2(s)}{Y_1(s)} = b_N s^N + \cdots + b_1 s + b_0
\]

\[X(s) = Y_0(s) = H_1(s) H_2(s) = \frac{b_N s^N + \cdots + b_1 s + b_0}{s^N + a_N - \cdots + a_1 s + a_0} \rightarrow Y_0(s) = Y(s) = Y_1(s) Y_2(s) = Y_1(s) \cdot b_N s^N + \cdots + b_1 s + b_0\]
Standard Realizations of Systems

Cascade Form

The transfer function can be factored into the form,

\[ H(s) = \frac{s^2 - z_1}{s - p_1} \cdot \frac{s - z_2}{s - p_2} \cdot \frac{1}{s - p_3} \cdot \frac{1}{s - p_N} \]

and each factor can be realized in a small canonical-form subsystem of either of the two forms,

\[ H(s) = \frac{1}{s - p_i} \quad \text{or} \quad H(s) = \frac{1}{s - jq} \]

and these subsystems can then be cascade connected.

Parallel Form

The transfer function can be expanded in partial fractions of the form,

\[ H(s) = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_D}{s - p_D} \]

Each of these terms describes a subsystem. When all the subsystems are connected in parallel the overall system is realized.