Description of Systems

- Broadly speaking, a system is anything that responds when stimulated or excited.
- The systems most commonly analyzed by engineers are artificial systems designed and built by humans.
- Engineering system analysis is the application of mathematical methods to the design and analysis of systems.

System Examples

Feedback Systems

- In a feedback system the response of the system is "fed back" and combined with the excitation in such a way as to optimize the response in some desired sense.
- Examples of feedback systems:
  1. Temperature control in a house using a thermostat.
  2. Water level control in the tank of a flush toilet.
  3. Pouring a glass of lemonade to the top of the glass without overflowing.
  4. A refrigerator ice maker which keeps the bin full of ice but does not make extra ice.
  5. Driving a car.

Systems

- Systems have inputs and outputs.
- Systems accept excitations or input signals at their inputs and produce responses or output signals at their outputs.
- Systems are often usefully represented by block diagrams.

A single-input, single-output system block diagram:

\[ x(t) \xrightarrow{H} y(t) \]
Block Diagram Symbols

Three common block diagram symbols for an amplifier (we will use the last one).

Three common block diagram symbols for a summing junction (we will use the first one).

An Electrical Circuit Viewed as a System

An RC lowpass filter is a simple electrical system. It is excited by a voltage \( v_i(t) \) and responds with a voltage \( v_o(t) \). It can be viewed or modeled as a single-input, single-output system.

Zero-State Response of an RC Lowpass Filter to a Step Excitation

If an RC lowpass filter with an initially uncharged capacitor is excited by a step of voltage \( v_i(t) = A \delta(t) \) its response is \( v_o(t) = A(1 - e^{-t/RC}) u(t) \). This response is called the zero-state response of this system because there was initially no energy stored in the system. (It was in its zero-energy state.) If the excitation is doubled, the zero-state response also doubles.

Zero-Input Response of an RC Lowpass Filter

If an RC lowpass filter has an initial charge on the capacitor of \( V_0 \) volts and no excitation is applied to the system its zero-input response is \( v_o(t) = V_0 e^{-t/RC}, t > 0 \).

Homogeneity

In a homogeneous system, multiplying the excitation by any constant (including complex constants), multiplies the zero-state response by the same constant.
Homogeneity

To test a system for **homogeneity** use this logical process. Apply a signal $g(t)$ as the excitation $x_i(t)$ and find the zero-state response $y_i(t)$. Then apply the signal $Kg(t)$ as $x_i(t)$ where $K$ is a constant and find the zero-state response $y_2(t)$. If $y_2(t) = Ky_1(t)$ for any arbitrary $g(t)$ and $K$, then the system is homogeneous.

If $g(t) \rightarrow y_1(t)$ and $Kg(t) \rightarrow Ky_1(t)$

$\mathcal{H}$ is Homogeneous

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Time Invariance

- If an excitation causes a zero-state response and delaying the excitation simply delays the zero-state response by the same amount of time, regardless of the amount of delay, the system is **time invariant**.

**Time Invariant System**

\[ x(t) \xymatrix{ & \rightarrow \mathcal{H} & y(t) \ar[l] } \]

\[ x(t) \rightarrow \mathcal{H} \rightarrow y(t) \]

\[ x(t) \rightarrow \mathcal{H} \rightarrow y(t) \]

\[ y(t-t_0) \]

If $g(t) \rightarrow y_1(t)$ and $g(t-t_0) \rightarrow y_2(t-t_0)$ \( \Rightarrow \mathcal{H} \) is Time Invariant

This test must succeed for any $g$ and any $t_0$. 

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Additivity

- If an excitation causes a zero-state response and another excitation causes another zero-state response and if, for any arbitrary excitations, the sum of the two excitations causes a zero-state response that is the sum of the two zero-state responses, the system is said to be **additive**.

**Additive System**

\[ x(t) \xymatrix{ & \rightarrow \mathcal{H} & y(t) \ar[l] } \]

\[ x(t) \rightarrow \mathcal{H} \rightarrow y(t) \]

\[ x(t) \rightarrow \mathcal{H} \rightarrow y(t) \]

\[ y(t-t_0) \]

\[ y(t) + y(t) \rightarrow y(t) + y(t) \]

\[ y(t) + y(t) \rightarrow y(t) + y(t) \]

If $g(t) \rightarrow y_1(t)$ and $h(t) \rightarrow y_2(t)$

and $g(t) + h(t) \rightarrow y_3(t)$ and $y_3(t) = y_1(t) + y_2(t)$ \( \Rightarrow \mathcal{H} \) is Additive

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Homogeneity

Let $y(t) = \exp\{x(t)\}$. Is this system homogeneous?

Let $x(t) = g(t)$. Then $y(t) = \exp\{g(t)\}$.

Let $x(t) = Kg(t)$. Then $y(t) = \exp\{Kg(t)\} = [\exp\{g(t)\}]^K$

$Ky(t) = K\exp\{g(t)\} \Rightarrow y(t) \neq Ky(t)$, Not homogeneous

Let $y(t) = x(t)+2$. Is this system homogeneous?

Let $x(t) = g(t)$. Then $y(t) = g(t)+2$.

Let $x(t) = Kg(t)$. Then $y(t) = Kg(t)+2$

$Ky(t) = Kg(t)+2K \Rightarrow y(t) \neq Ky(t)$, Not homogeneous

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Time Invariance

Let $y(t) = \exp\{x(t)\}$. Is this system time invariant?

Let $x(t) = g(t)$. Then $y(t) = \exp\{g(t)\}$.

Let $x(t) = g(t-t_0)$. Then $y(t) = \exp\{g(t-t_0)\}$

$y(t-t_0) = \exp\{g(t-t_0)\} = y(t) \Leftrightarrow y(t) = y(t-t_0)$, Time Invariant

Let $y(t) = x(t)/2$. Is this system time invariant?

Let $x(t) = g(t)$. Then $y(t) = g(t)/2$.

Let $x(t) = g(t-t_0)$. Then $y(t) = g(t-t_0)/2$

$y(t-t_0) = g(t-t_0)/2 \Rightarrow y(t) \neq y(t-t_0)$, Time Variant

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Additivity

Let $y(t) = u\{x(t)\}$. Is this system additive?

Let $x(t) = g(t)$. Then $y(t) = u\{g(t)\}$.

Let $x(t) = h(t)$. Then $y(t) = u\{h(t)\}$.

Let $x(t) = g(t) + h(t)$. Then $y(t) = u\{g(t) + h(t)\}$

$y(t) + y(t) = u\{g(t)\} + u\{h(t)\} \neq u\{g(t) + h(t)\}$, Not additive.

(For example, at time $t=3$, if $g(3) = 4$ and $h(3) = 2$, $y(3) + y(3) = u(4) + u(2) = 1+1 = 2$. But $y(3) = u(4+6) = 1$.)
Linearity and LTI Systems

- If a system is both homogeneous and additive it is **linear**.
- If a system is both linear and time-invariant it is called an **LTI** system.
- Some systems which are non-linear can be accurately approximated for analytical purposes by a linear system for small excitations.

**Stability**

- Any system for which the response is bounded for any arbitrary bounded excitation, is called a **bounded-input-bounded-output (BIBO)** stable system.
- A continuous-time LTI system described by a differential equation is stable if the **eigenvalues** of the solution of the equation **all** have negative real parts.

**Causality**

- Any system for which the zero-state response occurs only during or after the time in which the excitation is applied is called a **causal** system.
- Strictly speaking, all real physical systems are causal.

**Memory**

- If a system’s zero-state response at any arbitrary time depends only on the excitation at that same time and not on the excitation or response at any other time it is called a **static** system and is said to have no **memory**. All static systems are causal.
- A system whose zero-state response at some arbitrary time depends on anything other than the excitation at that same time is called a **dynamic** system and is said to have memory.
- Any system containing an integrator has memory.

**Static Non-Linearity**

- Many real systems are non-linear because the relationship between excitation amplitude and response amplitude is non-linear.
Static Non-Linearity

- For an analog multiplier, if the two excitations are the same single excitation signal, the response signal is the square of that single excitation signal and doubling the excitation would cause the response to increase by a factor of 4.
- Such a system is not homogeneous and therefore not linear.

Invertibility

A system is said to be invertible if unique excitations produce unique zero-state responses. In other words, if a system is invertible, knowledge of the zero-state response is sufficient to determine the excitation.

Dynamics of Second-Order Systems

The eigenfunction of an LTI system is the complex exponential. The eigenvalues of a second-order system are either both real or occur in a complex-conjugate pair. The general solution form is a sum of two complex exponentials and a constant. For example, the capacitor voltage in a series RLC circuit excited by a voltage step of height \( A \) is

\[
v_c(t) = K_e^{-\omega_L t} + K_r e^{j\omega_c t} + A
\]

where \( \alpha = R/2L \) and \( \omega_0^2 = 1/\text{LC} \). \( \alpha \) is the damping factor and \( \omega_0 \) is the natural radian frequency.

Complex Sinusoid Excitation

Any LTI system excited by a complex sinusoid responds with another complex sinusoid of the same frequency but generally a different magnitude and phase. In the case of the RLC circuit if the excitation is \( v_e(t) = A e^{j\omega t} \) the response is \( v_r(t) = Be^{j\omega t} \), where \( A \) and \( B \) are, in general, complex. \( B \) can be found by substituting the solution form into the differential equation and finding the particular solution. In the RLC circuit

\[
B = \frac{A}{j2\pi f_0} \frac{1}{LC + j2\pi f_0 RC + 1}
\]

Discrete-Time Systems

- With the increase in speed and decrease in cost of digital system components, discrete-time systems have experienced, and are still experiencing, rapid growth in modern engineering system design.
- Discrete-time systems are usually described by difference equations.
Block Diagram Symbols

The block diagram symbols for a summing junction and an amplifier are the same for discrete-time systems as they are for continuous-time systems.

Block diagram symbol for a delay

$$x[n] \rightarrow D \rightarrow x[n - 1]$$

Discrete-Time Systems

In a discrete-time system events occur at points in time but not between those points. The most important example is a digital computer. Significant events occur at the end of each clock cycle and nothing of significance (to the computer user) happens between those points in time.

Discrete-time systems can be described by difference (not differential) equations. Let a discrete-time system generate an excitation signal \(y[n]\) where \(n\) is the number of discrete-time intervals that have elapsed since some beginning time \(n = 0\). Then, for example a simple discrete-time system might be described by

\[
y[n] = 1.97y[n-1] - y[n-2]
\]

Discrete-Time Systems

The equation

\[
y[n] = 1.97y[n-1] - y[n-2]
\]

says in words

“The signal value at any time \(n\) is 1.97 times the signal value at the previous time \((n-1)\) minus the signal value at the time before that \((n-2)\)”

If we know the signal value at any two times, we can compute its value at all other (discrete) times. This is quite similar to a second-order differential equation for which knowledge of two independent initial conditions allows us to find the solution for all time and the solution methods are very similar.

Discrete-Time Systems

We could solve this equation by iteration using a computer.

\[
y[n] = 1 \quad ; \quad y[1] = 0; \quad \text{--- Initial Conditions}
\]

while 1,

\[
y[n] = y[n-1] + y[n-2] + 1.97 \cdot y[n-1] - y[n-2];
\]

end

We could also describe the system with a block diagram.

Solving Difference Equations

In the previous two slides we found the solution to

\[
y[n] = 1.97y[n-1] - y[n-2]
\]

by iteration as a sequence of numbers for \(y[n]\). We can also solve linear, constant-coefficient ordinary difference equations with techniques that are very similar to those used to solve linear, constant-coefficient ordinary differential equations. The eigenfunction of this type of equation is the complex exponential \(z^n\). As a first example let the equation be

\[
y[n] - y[n-1] = 0.
\]

The homogeneous solution of this equation is

\[
y_h[n] = K z^n.
\]

Substituting that into the equation we get

\[
2Kz^n - Kz^{n-1} = 0.
\]

This is the characteristic equation. Dividing through by \(Kz^{n-1}\) we get \(2z - 1 = 0\) and the solution is \(z = 1/2\).
**Solving Difference Equations**

The eigenvalue for the equation $2y[n] - y[n-1] = 0$ is then $\lambda = 1/2$ and the homogeneous solution is $y_h[n] = K(1/2)^n$. Since the equation is homogeneous, the homogeneous solution is also the total solution.

To find $K$ we need an initial condition. Let it be $y[0] = 3$. Then $y[0] = K(1/2)^0 = K = 3$ and $y[n] = 3(1/2)^n$.

**System Properties**

- The properties of discrete-time systems have the same meaning as they do in continuous-time systems.

If the excitation is doubled, the zero-state response doubles. If two signals are added to form the excitation, the zero-state response is the sum of the zero-state responses to those two signals. If the excitation is delayed by some time, the zero-state response is delayed by the same time. This system is linear and time invariant.
Eigenfunctions of LTI Systems

- The eigenfunction of an LTI system is the complex exponential
- The eigenvalues are either real or, if complex, occur in complex conjugate pairs
- Any LTI system excited by a complex sinusoid responds with another complex sinusoid of the same frequency, but generally a different amplitude and phase
- All these statements are true of both continuous-time and discrete-time systems